

Technical Report No. 32-197

**Radiation Exchange Between Two Flat Surfaces
Separated by an Absorbing Gas**

Robert V. Meghreblían



**JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA**

April 6, 1962

**CASE FILE
COPY**

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CONTRACT No. NAS 7-100

Technical Report No. 32-197

**Radiation Exchange Between Two Flat Surfaces
Separated by an Absorbing Gas**

Robert V. Meghreblían


Robert V. Meghreblían, Chief
Physical Sciences Division

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

April 6, 1962

Copyright© 1962
Jet Propulsion Laboratory
California Institute of Technology

CONTENTS

I.	Introduction	1
II.	Absorbing Gas at Uniform Temperature	3
	A. Radiation from Gas	3
	B. Net Radiation from Wall 1 plus Gas to Wall 2	5
	C. Net Radiation from Wall 1 to Wall 2	6
	D. Limiting Cases	7
III.	Absorbing Gas with Equilibrium Temperature Profile	9
	A. Temperature Distribution	9
	B. Net Radiative Transfer	12
Appendix.	The Asymmetry Property of the Function $\theta(x)$	16
References	18

FIGURES

1.	Approximate solution for temperature distribution function	13
2.	Net radiative transfer to cold wall	15

12150

ABSTRACT

An approximate analytical solution is obtained for the temperature distribution in an absorbing gas layer bounded by two flat surfaces radiating at different temperatures. The corresponding expression for the net radiation exchange between the two surfaces is derived and numerical solutions obtained for gas layers up to 10 optical thicknesses in depth. Similar results are derived for the special case of a gas layer at uniform temperature, and the net radiation exchange in this system is found to agree well with that in the previous for gas layers less than two optical thicknesses in depth.

I. INTRODUCTION

A problem in radiative transfer of increasing importance to the calculation of the thermal balance in high-temperature propulsion devices is the net radiation exchange between two surfaces at different temperatures separated by an absorbing gas. Various specialized aspects of this general problem have received the attention of investigators from time to time during the past century. Some of the methods used in these analyses have been summarized by Jakob (1). Of particular interest are the analytical methods of Wohlenberg (2) and Christianson (3) for obtaining the net radiative transfer in systems wherein the absorbing gas is at a uniform temperature. More recently, the detailed problem of determining the steady-state temperature distribution in the gas, as well as the net energy transfer, has been treated by Usiskin and Sparrow (4) who obtained numerical integrations of the appropriate integral equations.

The present treatment examines some of these same aspects of the problem by means of a dual approach. In the first portion of the analysis, the methods of Wohlenberg and Christianson are combined so as to handle the case of two grey surfaces separated by an absorbing gas at uniform temperature. In the second portion, the problem considered by Usiskin and Sparrow is reviewed, and an approximate analytical solution is derived which yields closed-form expressions for the temperature distribution in the gas and the net radiative transfer across the system in terms of the relevant physical parameters.

II. ABSORBING GAS AT UNIFORM TEMPERATURE

The problem considered here is that of determining the radiation exchange between two flat grey surfaces separated by an absorbing gas at uniform temperature. The absolute temperature of one surface is T_1 and of the other, T_2 , where $T_1 > T_2$. It is further specified that the surface at temperature T_i has emissivity ϵ_i , and that this is equal to its absorptivity. The gas has absorptivity α_i to the radiation emitted from the surface at temperature T_i . The radiation balance relations are stated in terms of three fundamental quantities:

$$W_i \equiv \sigma \epsilon_i T_i^4 = \text{grey-body radiation (energy/area) from surface at temperature } T_i; \quad (1)$$

σ is the Stephan – Boltzmann constant.

$$W_{ij} \equiv \text{radiation/area from medium } i \text{ absorbed by medium } j. \quad (2)$$

$$q_{i,j} \equiv \text{direct radiative transfer/area between media } i \text{ and } j. \quad (3)$$

A. Radiation from Gas

The first step in the analysis is the derivation of an expression for the radiation from the gas. For this purpose we require such quantities as W_{g2} , the radiation emitted by the gas which is absorbed by the wall at temperature T_2 . The quantity W_{g2} consists of two parts, that portion of the gas radiation which goes directly to wall 2, and that which goes first to wall 1 and, after reflection, is eventually absorbed by wall 2. The application of Christianson's method of multiple reflections yields for the former:

$$W_g \left\{ \epsilon_2 + (1 - \epsilon_2)(1 - \alpha_2)(1 - \epsilon_1)(1 - \alpha_1) \epsilon_2 \right. \\ \left. + [(1 - \epsilon_2)(1 - \alpha_2)(1 - \epsilon_1)(1 - \alpha_1)]^2 \epsilon_2 + \dots \right\} = \frac{\epsilon_2 W_g}{1 - \eta}$$

where

$$\eta \equiv (1 - \alpha_1)(1 - \alpha_2)(1 - \epsilon_1)(1 - \epsilon_2) \quad (4)$$

is the fraction of the radiation leaving either solid surface which is reflected once by each; the last equality in the above equation follows by virtue of the fact that $\eta < 1$. By applying this approach also to the radiation component which goes first to wall 1, is reflected and subsequently absorbed by wall 2, one obtains

$$\frac{\epsilon_2 W_g (1 - \epsilon_1)(1 - \alpha_1)}{(1 - \eta)}$$

Adding these expressions yields W_{g2} :

$$W_{g2} = \frac{\epsilon_2 W_g}{1 - \eta} [1 + (1 - \epsilon_1)(1 - \alpha_1)] \quad (5)$$

A related quantity of interest is W_{2g} , the radiation emitted by wall 2 which is ultimately absorbed by the gas. By a similar line of reasoning to that applied above, it is easily shown that

$$W_{2g} = \frac{W_2}{1 - \eta} [\alpha_2 + \alpha_1 (1 - \epsilon_1)(1 - \alpha_2)] \quad (6)$$

It is of interest to mention that the radiation emitted by wall 2 and subsequently reabsorbed, W_{22} , is given by

$$W_{22} = \frac{W_2 (1 - \alpha_1)(1 - \alpha_2)(1 - \epsilon_1) \epsilon_2}{1 - \eta} \quad (7)$$

and, of course,

$$W_2 = W_{22} + W_{21} + W_{2g} \quad (8)$$

Corresponding expressions for W_{g1} and W_{1g} are obtained by simply exchanging indices in the above relations.

An explicit solution for W_g (which yields the equilibrium temperature of the gas in this system) is obtained by requiring that the net radiation passing from wall 1 to gas is equal to that from gas to wall 2. Using the notation of definition (3), the latter is given by

$$q_{g,2} = W_{g2} - W_{2g} = \frac{\epsilon_2 W_g [1 + (1 - \epsilon_1)(1 - \alpha_1)] - W_2 [\alpha_2 + \alpha_1 (1 - \epsilon_1)(1 - \alpha_2)]}{1 - \eta} \quad (9)$$

and the former by $-q_{g,1}$, which may be obtained immediately from (9) by exchanging indices. Now, at steady state,

$$-q_{g,1} = q_{g,2} \quad (10)$$

from which follows

$$W_g = \frac{W_2 [\alpha_2 + \alpha_1 (1 - \epsilon_1)(1 - \alpha_2)] + W_1 [\alpha_1 + \alpha_2 (1 - \epsilon_2)(1 - \alpha_1)]}{\epsilon_2 [1 + (1 - \epsilon_1)(1 - \alpha_1)] + \epsilon_1 [1 + (1 - \epsilon_2)(1 - \alpha_2)]} \quad (11)$$

B. Net Radiation from Wall 1 plus Gas to Wall 2

The combined net radiation from wall 1 and gas to wall 2 is given by

$$q_{1+g,2} = W_{12} + W_{g2} - (W_{21} + W_{2g}) \quad (12)$$

where W_{g2} and W_{2g} are obtained from Eqs. (5) and (6), and

$$W_{12} = \frac{W_1 (1 - \alpha_1) \epsilon_2}{1 - \eta} \qquad W_{21} = \frac{W_2 (1 - \alpha_2) \epsilon_1}{1 - \eta} \qquad (13)$$

Substitution of these expressions into Eq. (12) yields

$$q_{1+g,2} = \frac{1}{1 - \eta} \{ W_1 (1 - \alpha_1) \epsilon_2 + \epsilon_2 W_g [1 + (1 - \epsilon_1)(1 - \alpha_1)] \\ - W_2 [(1 - \alpha_2) \epsilon_1 + \alpha_2 + \alpha_1 (1 - \epsilon_1)(1 - \alpha_2)] \} \qquad (14)$$

C. Direct Radiative Exchange Between Wall 1 and Wall 2

The direct radiative exchange between walls 1 and 2 is obtained immediately from results (9) and (14); thus,

$$q_{1,2} = q_{1+g,2} - q_{g,2} = W_{12} - W_{21} \qquad (15)$$

Substitution from Eq. (13) yields

$$q_{1,2} = \frac{W_1 \epsilon_2 (1 - \alpha_1) - W_2 \epsilon_1 (1 - \alpha_2)}{1 - \eta} \qquad (16)$$

D. Limiting Cases

Three limiting cases for the above results are of particular practical importance.

Consider first the case when both surfaces are black-body radiators; then $\epsilon_1 = \epsilon_2 = 1$. The corresponding expression for the radiation from the gas is obtained from Eq. (11).

$$W_g = \frac{1}{2} (\alpha_2 W_2 + \alpha_1 W_1) \quad (17)$$

The net radiation exchange between wall 1 and gas and wall 2 then reduces to

$$q_{1+g,2} = (1 - \alpha_1) W_1 + W_g - W_2 \quad (18)$$

which is in agreement with the result given by Jakob.* The substitution of the form (17) into (18) yields

$$q_{1+g,2} = (1 - \frac{1}{2} \alpha_1) W_1 - (1 - \frac{1}{2} \alpha_2) W_2 \quad (19)$$

Finally, one can also obtain the appropriate result for the net exchange between walls 1 and 2 from (16).

$$q_{1,2} = (1 - \alpha_1) W_1 - (1 - \alpha_2) W_2 \quad (20)$$

* Ref. 1, Vol. II, p. 132.

The second limiting case of interest is that of an entirely opaque gas, that is, $\alpha_1 = \alpha_2 = 1$. The three principal results in this case reduce to

$$W_g = \frac{W_2 + W_1}{\epsilon_1 + \epsilon_2} \quad (21)$$

$$q_{1+g,2} = \epsilon_2 W_g - W_2 = \frac{\sigma \epsilon_1 \epsilon_2 (T_1^4 - T_2^4)}{\epsilon_1 + \epsilon_2} \quad (22)$$

where the last equality is obtained by the application of Eq. (1) and (21). The direct exchange between walls 1 and 2 is then

$$q_{1,2} = 0 \quad (23)$$

as expected, since the gas layer completely shields one wall from the other.

The last special case is that of a completely transparent gas, that is, $\alpha_1 = \alpha_2 = 0$. Then,

$$W_g = 0 \quad (24)$$

$$q_{1+g,2} = \frac{\sigma \epsilon_1 \epsilon_2 (T_1^4 - T_2^4)}{1 - (1 - \epsilon_1)(1 - \epsilon_2)} \quad (25)$$

again using Eq. (1). This last expression is the familiar form reported in the literature (see Refs. 5 and 6).

The corresponding expression for $q_{1,2}$ is identical to Eq. (25), as is to be expected, since the gas layer cannot influence the energy balance by virtue of its transparency.

III. ABSORBING GAS WITH EQUILIBRIUM TEMPERATURE PROFILE

The simplified physical problem treated in the preceding section is now generalized to account for the presence of the gas layer as an emitting and absorbing medium whose internal temperature at any local station is determined by the net energy exchange with all other radiating elements in the system. In carrying out the analysis we introduce the following simplifying assumptions: (1) the two solid boundaries radiate as black bodies; (2) the gas radiates as a grey body; and (3) heat transfer by conduction is negligible.

An immediate consequence of assumption (2) is that the absorption characteristics of the gas layer may be represented by a single coefficient of absorption (i.e., the logarithmic decrement of radiation) which is independent of wavelength. For further simplification, we assume that this coefficient is also independent of temperature. By virtue of assumption (3), the energy balance in the system may be specified entirely in terms of the radiative balance. The appropriate equation is obtained from an analytical statement of the energy balance in a differential volume element in the gas and yields a linear integral equation for the temperature distribution within the gas. This equation has been derived by Usiskin and Sparrow (4) and is given as Eq. (15) in the reference paper.

A. Temperature Distribution

The purpose of the present section is to develop an approximate analytical solution for the "temperature" (actually radiative power) distribution which involves directly the essential physical parameters of the system, and subsequently to apply this result to the calculation of the net radiative transfer across the gas gap. The governing energy balance equation of Usiskin and Sparrow is taken over directly for this analysis, and it may be shown, upon introducing the generalized exponential integrals and some of their properties, that this equation may be written in the following form

$$2W_g(x) = \frac{S(x)}{2k} + W_2 E_2(\kappa x) + W_1 E_2(\kappa(1-x)) + \kappa \int_0^x W_g(s) E_1(\kappa(x-s)) ds + \kappa \int_x^1 W_g(s) E_1(\kappa(s-x)) ds \quad (26)$$

where $0 \leq x \leq 1$ is the fractional distance across the gas layer measured from the cold (T_2) wall. The quantity $4\kappa W_g(x) dx$ represents the energy radiated per unit time from a gas layer of thickness dx and unit

cross-sectional area (2). A special case of $W_g(x)$ is the familiar expression $\sigma T_g^4(x)$, where $T_g(x)$ is the temperature of the gas. The symbol W_i is defined as in the preceding analysis. The function $S(x)$ represents the internal energy sources in the gas (with units of power/volume); the parameter κ is the optical thickness of the gas layer and is defined $\kappa \equiv kL$; k is the absorption coefficient of the gas and L is its thickness. The $E_n(x)$ are the generalized exponential integrals; these are defined (7),

$$E_n(x) \equiv x^{n-1} \int_x^\infty \frac{e^{-s} ds}{s^n}$$

Equation (26) states simply that the power radiated from a unit volume of gas at x (the term on the left-hand side) is equal, at steady state, to the radiated power it absorbs from all other elements. The first term on the right-hand side accounts for the internal sources; the second term gives the portion of the radiation emitted by the cold wall which is absorbed by the gas volume, and the third gives that portion from the hot wall. The third and fourth terms account for radiation from the rest of the gas layer which is absorbed in the unit volume about x .

This energy balance equation may be considerably simplified by introducing the dimensionless function $\theta(x)$:

$$\theta(x) \equiv \frac{W_g(x) - \sigma T_2^4}{\sigma(T_1^4 - T_2^4)} \quad (27)$$

The substitution of this function into Eq. (26) yields

$$2\theta(x) = \frac{S(x)}{2k\sigma(T_1^4 - T_2^4)} + E_2(\kappa(1-x)) + \kappa \int_0^1 \theta(s) E_1(\kappa|x-s|) ds \quad (28)$$

where we have used the result

$$\int_0^x E_1(s) ds = 1 - E_2(x)$$

In this analysis attention is focussed on the solution to Eq. (28) for the case of no internal heat sources in the gas; thus, $S(x) = 0$. The resulting equation is in the form of the inhomogeneous Fredholm equation of the second kind, and formal solutions can be generated by, for example, a Liouville-Neumann series expansion (8). An examination of the numerical results obtained by Usiskin and Sparrow reveals, however, that the "temperature" function $\theta(x)$ has nearly a linear dependency on x , especially for large κ . Therefore, in constructing our approximate analytical solution we introduce the form $\theta(x; \kappa) = a(\kappa)x + c(\kappa)$ into the integral as a zeroth-order approximation and calculate the resulting expression for $\theta(x; \kappa)$ at the left. If the assumed linear form is indeed a reasonable first estimate for $\theta(x)$, then the resulting expression from the left should have the form

$$\theta(x; \kappa) = a(\kappa)x + c(\kappa) + R(x; \kappa) \quad (29)$$

where the correction term $R(x; \kappa)$ will be small.

For computational purposes, it is convenient to write the source-free equation in the form

$$2\theta(x) = E_2(\kappa(1-x)) + \kappa \int_0^x \theta(s) E_1(\kappa(x-s)) ds + \kappa \int_x^1 \theta(s) E_1(\kappa(s-x)) ds \quad (30)$$

The application of the procedure mentioned above is straightforward, and leads rather easily to the form (29). In carrying out this calculation it is necessary to impose also two conditions on $\theta(x)$ so that the specification of the quantities $a(\kappa)$ and $c(\kappa)$ is complete. These are

$$\theta\left(\frac{1}{2}; \kappa\right) = \frac{1}{2} \quad \theta'\left(\frac{1}{2}; \kappa\right) = a(\kappa) \quad (31)$$

The first condition is a consequence of the antisymmetry property of the function θ (see Appendix for proof). The second condition implies that $R'(1/2; \kappa) = 0$. The substitution of the linear form for θ into the right-hand side of Eq. (30) yields the result:

$$R(x; \kappa) = \frac{1}{2} \left\{ \frac{a(\kappa)}{2\kappa} \left[e^{-\kappa x} - e^{-\kappa(1-x)} \right] - \left[c(\kappa) + \frac{1}{2} a(\kappa)x \right] E_2(\kappa x) + \left[1 - c(\kappa) - \frac{1}{2} a(\kappa)(1+x) \right] E_2(\kappa(1-x)) \right\} \quad (32)$$

In performing the calculation the following property of the $E_n(x)$ is required:

$$\int_0^x s E_1(s) ds = \frac{1}{2} - x E_2(x) - E_3(x)$$

The application of relations (31) yields

$$a(\kappa) = \frac{\kappa}{2} e^{\kappa/2} E_1\left(\frac{\kappa}{2}\right)$$

$$c(\kappa) = \frac{1}{2} [1 - a(\kappa)]$$
(33)

The resulting expression for $\theta(x; \kappa)$ given in Eq. (29) (see Fig. 1) when compared graphically with the numerical solution obtained by Usiskin and Sparrow is found to be accurate to less than a percent.

The comparison is especially good for large κ ; when $\kappa \gg 1$, the expressions for a and c have the asymptotic forms:

$$a(\kappa) \sim 1 - \frac{2}{\kappa} \qquad c(\kappa) \sim \frac{1}{\kappa}$$
(34)

It may be observed from the form of the function (29) that further iteration will yield an integral equation for the correction term $R(x; \kappa)$. Because of the antisymmetry property of θ , the next-higher-order approximation may be constructed by introducing an x^3 term into (29). A continuation of this procedure will generate a power series in odd x which will converge, thereby providing an increasingly accurate analytical expression for θ .

B. Net Radiative Transfer

Of practical interest is the calculation of the net radiative transfer between the hot wall and gas and the cold wall. This corresponds to the quantity $q_{1+g,2}$ determined in the previous section for the case

of an isothermal gas [see Eq. (14)]. Again, the appropriate expression has been derived by Usiskin and Sparrow.

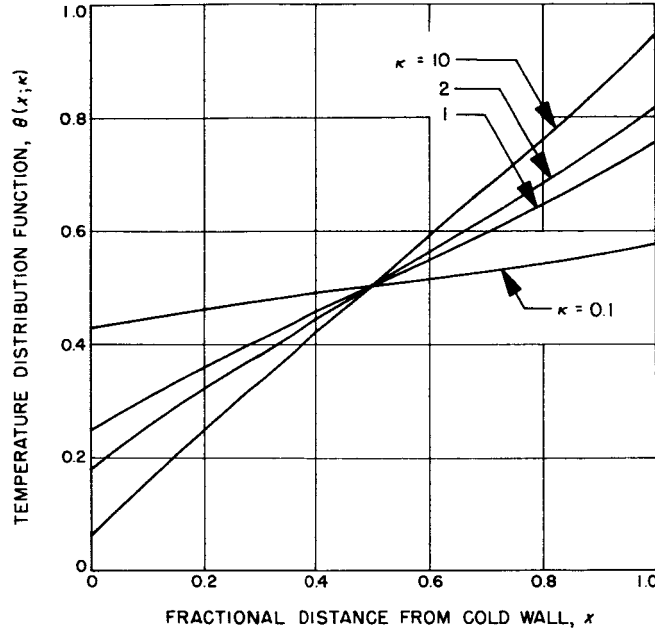


Fig. 1. Approximate solution for temperature distribution function

By introducing the $E_n(x)$ functions and combining various terms, their result may be written in the form

$$Q(\kappa) = 2E_3(\kappa) + 2\kappa \int_0^1 \theta(s) E_2(\kappa s) ds \quad (35)$$

where

$$Q(\kappa) \equiv \frac{q(\kappa)}{\sigma(T_1^4 - T_2^4)} \quad (36)$$

and $q(\kappa)$ is the net energy transferred per unit area and time to the cold wall. It is easily shown that $Q(\kappa)$ has the following properties:

$$\lim_{\kappa \rightarrow 0} Q(\kappa) = 1 \qquad \lim_{\kappa \rightarrow \infty} Q(\kappa) = 0 \quad (37)$$

as is to be expected on physical grounds. The function $Q(\kappa)$ has been computed by Usiskin and Sparrow for the interval $(0 \leq \kappa \leq 2)$ using the numerical solution for $\theta(x)$ mentioned previously. Of particular interest to this discussion is the comparison of their result with that obtained using the analytical form for θ suggested in the preceding section. If one neglects entirely the correction term R , then direct substitution into Eq. (35) yields

$$Q(\kappa) = \frac{1}{2} \left[1 + \left(\frac{4}{3\kappa} - 1 \right) a(\kappa) \right] - \frac{2a(\kappa)e^{-\kappa}}{3\kappa} + \left[1 - \frac{a(\kappa)}{3} \right] E_3(\kappa) \quad (38)$$

It may be shown that this result also satisfies the limits (37). A graphical comparison with the exact numerical solution is shown in Fig. 2. Evidently the use of the linear form for θ yields an estimate for Q which agrees to within about 10 percent in the interval $(0.1 < \kappa < 1)$; even better agreement is obtained outside the interval.

A final comparison of some practical value may be drawn from the result obtained for the net radiation exchange $Q(\kappa)$ using the relations derived for the uniform temperature gas system discussed in the preceding section. The appropriate expression is given in Eq. (19), which for the present system may be written

$$Q(\kappa) \equiv \frac{q_{1+g,2}}{W_1 - W_2} = 1 - \frac{1}{2} \alpha(\kappa)$$

where $\alpha_1 = \alpha_2 = \alpha$. From the definitions of α and k , it follows that

$$\alpha(\kappa) = 1 - e^{-\kappa} \quad (39)$$

Thus, for the uniform temperature gas,

$$Q(\kappa) = \frac{1}{2} (1 + e^{-\kappa}) \quad (40)$$

This function is shown in Fig. 2, and the agreement with the variable temperature case is reasonably good up to two optical thicknesses of gas layer.

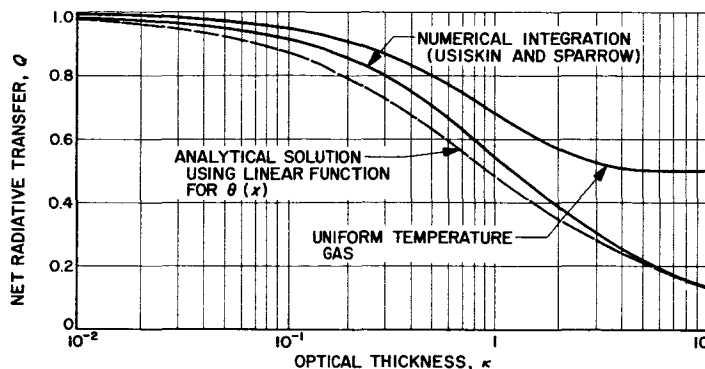


Fig. 2. Net radiative transfer to cold wall

It is interesting to note the difference in the limits for the two cases when $\kappa \rightarrow \infty$. The result for the case with non-uniform gas temperature was discussed previously [see Eq. (37)]. The result for the uniform temperature gas case is also explainable on physical grounds. In the limit of a very opaque gas, it is to be expected that the net radiative transfer to the cold wall will be simply the difference between the radiation emitted by the gas and the cold wall; thus, one can write immediately

$$\lim_{\kappa \rightarrow \infty} Q(\kappa) = \frac{W_g(\infty) - W_2}{W_1 - W_2} \quad (41)$$

where $W_g(\infty)$ denotes the radiation emitted by an opaque gas (i.e., $\alpha \rightarrow 1$, $\kappa \rightarrow \infty$). If the expression (21) is used for this quantity, along with $\epsilon_1 = 1$, then it follows that $Q(\kappa) \rightarrow 1/2$ as $\kappa \rightarrow \infty$.

APPENDIX. The Antisymmetry Property* of the Function $\theta(x)$.

Consider the function $\theta(x)$ which satisfies the integral equation [cf. Eq. (30)]

$$2\theta(x) = E_2(\kappa(1-x)) + \kappa \int_0^1 \theta(s) E_1(\kappa|x-s|) ds \quad 0 \leq x \leq 1 \quad (A-1)$$

Transform to the function $\phi(x)$, where

$$\theta(x) \equiv \phi(x) + \frac{1}{2} \quad (A-2)$$

Substitution into Eq. (A-1) yields

$$2\phi(x) + 1 = E_2(\kappa(1-x)) + \kappa \int_0^1 \phi(s) E_1(\kappa|x-s|) ds + \frac{\kappa}{2} \int_0^1 E_1(\kappa|x-s|) ds \quad (A-3)$$

also

$$2\phi(1-x) + 1 = E_2(\kappa x) + \kappa \int_0^1 \phi(1-s) E_1(\kappa|x-s|) ds + \frac{\kappa}{2} \int_0^1 E_1(\kappa|x-s|) ds \quad (A-4)$$

If a new function $f(x) \equiv \phi(x) + \phi(1-x)$ is introduced, then the addition of Eq. (A-3) and (A-4) reveals that $f(x)$ satisfies the integral equation

$$2f(x) = \kappa \int_0^1 f(s) E_1(\kappa|x-s|) ds \quad (A-5)$$

wherein use has been made of the relation

$$\kappa \int_0^1 E_1(\kappa|x-s|) ds = 2 - E_2(\kappa x) - E_2(\kappa(1-x)) \quad (A-6)$$

* The author is indebted to C. Solloway for assistance in demonstration of this proof.

Clearly, $f(x) = 0$ is a solution to Eq. (A-5). This yields the property

$$\phi(x) + \phi(1-x) = 0 \quad (\text{A-7})$$

which is in fact the statement of antisymmetry for the function $\phi(x)$; i.e., $\phi(x)$ is an antisymmetric function of the argument $[x - (1/2)]$. It is obvious then that $\theta(x)$ is antisymmetric about the point $(1/2, 1/2)$ in the θ, x -plane.

To prove that $f(x) = 0$ is a unique solution to Eq. (A-5), suppose that there is some non-zero solution $f(x)$. Then,

$$2|f(x)| \leq |f|_{\max} \kappa \int_0^1 E_1(\kappa|x-s|) ds$$

$$< r |f|_{\max}$$

where $r < 2$, which follows from Eq. (A-6). Furthermore,

$$|f(x)| < \xi |f|_{\max}$$

where $\xi \equiv r/2$ and $0 < \xi < 1$. Since this inequality holds for all x , it follows that

$$|f|_{\max} \leq \xi |f|_{\max} < |f|_{\max} \quad (\text{A-8})$$

which is a contradiction. Therefore $f(x) = 0$ is a unique solution, and $\phi(x)$, and likewise $\theta(x)$, are antisymmetric functions.

REFERENCES

1. Jakob, M., *Heat Transfer*, John Wiley and Sons, Inc., New York, Vol. I, 1949, Vol. II, 1957.
2. Wohlenberg, W. J., *Eng. Bull.* Purdue University, Vol. 24, No. 4a, Engineering Experiment Sta. Res. Ser. No. 75, August 1940.
3. Christianson, C., *Wiedemanns Ann.* 19, 267, p. 1883.
4. Usiskin, C. M. and E. M. Sparrow, *Int. J. Heat Mass Transfer*, 1, Pergamon Press, New York, 1960, pp. 28-36.
5. Jakob, M. and G. A. Hawkins, *Elements of Heat Transfer and Insulation*, 2nd Ed., John Wiley and Sons, New York, 1950.
6. Condon, E. U. and H. Odishaw, Editors, *Handbook of Physics*, McGraw-Hill Book Co., Inc., New York, 1958, p. 5-76.
7. Case, K. M., F. de Hoffman and G. Placzek, *Introduction to the Theory of Neutron Transport*, Vol. I, Los Alamos Scientific Laboratory, New Mexico, 1953 (available through U. S. Government Printing Office, Washington, D. C.)
8. Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*, Vol. I, McGraw-Hill Book Co., Inc., New York, 1953, Ch. 8.